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Super congruences for two Apéry-like sequences

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Abstract

For $n = 0, 1, 2, \dots$ let $T_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2$ and $S_n = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}$. Then $\{T_n\}$ and $\{S_n\}$ are Apéry-like numbers. In this paper we obtain some congruences and pose several challenging conjectures for sums involving $\{T_n\}$ or $\{S_n\}$.

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1. Introduction

In 1979, Apéry [1] introduced the Apéry numbers $\{A_n\}$ and $\{A'_n\}$ given by

$$A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 \quad \text{and} \quad A'_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}.$$

It is well known that (see [3])

$$(n+1)^3A_{n+1} = (2n+1)(17n(n+1)+5)A_n - n^3A_{n-1} \quad (n \ge 1),$$
$$(n+1)^2A'_{n+1} = (11n(n+1)+3)A'_n + n^2A'_{n-1} \quad (n \ge 1).$$

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. The first kind Apéry-like numbers $\{u_n\}$ satisfy

(1.1) $\quad u_0 = 1,\ u_1 = b,\ (n+1)^3u_{n+1} = (2n+1)(an(n+1)+b)u_n - cn^3u_{n-1} \quad (n \ge 1),$

where $a, b, c \in \mathbb{Z}$ and $c \ne 0$. Let

$$D_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k},$$
$$b_n = \sum_{k=0}^{[n/3]}\binom{2k}{k}\binom{3k}{k}\binom{n}{3k}\binom{n+k}{k}(-3)^{n-3k},$$
$$T_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2,$$

where $[x]$ is the greatest integer not exceeding x. Then $\{A_n\}$, $\{D_n\}$, $\{b_n\}$ and $\{T_n\}$ are first kind Apéry-like numbers with $(a, b, c) = (17, 5, 1), (10, 4, 64), (-7, -3, 81), (12, 4, 16)$, respectively. The numbers $\{D_n\}$ are called Domb numbers, and $\{b_n\}$ are called Almkvist-Zudilin numbers. For $\{A_n\}$, $\{D_n\}$, $\{b_n\}$ and $\{T_n\}$ see A005259, A002895, A125143, A290575 in Sloane's database "The On-Line Encyclopedia of Integer Sequences".

In 2009 Zagier [34] studied the second kind Apéry-like numbers $\{u_n\}$ given by

$$(1.2) \qquad u_0 = 1,\ u_1 = b \quad \text{and} \quad (n+1)^2 u_{n+1} = (an(n+1)+b)u_n - cn^2 u_{n-1}\ (n \ge 1),$$

where $a, b, c \in \mathbb{Z}$ and $c \ne 0$. See [5,34]. Let

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n},$$

$$S_n = \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{n}{2k} 4^{n-2k} = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k},$$

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}, \quad Q_n = \sum_{k=0}^{n} \binom{n}{k}(-8)^{n-k} f_k,$$

$$W_n = \sum_{k=0}^{[n/3]} \binom{2k}{k}\binom{3k}{k}\binom{n}{3k}(-3)^{n-3k}.$$

In [34] Zagier stated that $\{A'_n\}$, $\{f_n\}$, $\{S_n\}$, $\{a_n\}$, $\{Q_n\}$ and $\{W_n\}$ are second kind Apéry-like sequences with $(a, b, c) = (11, 3, -1), (7, 2, -8), (12, 4, 32), (10, 3, 9), (-17, -6, 72), (-9, -3, 27)$, respectively. The sequence $\{f_n\}$ is called Franel numbers since Franel [8] introduced it in 1894. In [11,25] the author systematically investigated identities and congruences for sums involving S_n. For $\{A'_n\}$, $\{f_n\}$, $\{S_n\}$, $\{a_n\}$, $\{Q_n\}$ and $\{W_n\}$ see A005258, A000172, A081085, A002893, A093388 and A291898 in Sloane's database "The On-Line Encyclopedia of Integer Sequences".

Apéry-like numbers have fascinating properties and they are concerned with Picard-Fuchs differential equation, modular forms, hypergeometric series, elliptic curves, series for $\frac{1}{\pi}$, supercongruences, binary quadratic forms, combinatorial identities, Bernoulli numbers and Euler numbers. See for example [4,5,6,10,13,16,18,24,27,29,30].

Throughout this paper, $H_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \in \mathbb{Z}^+$. For $a \in \mathbb{Z}$ and given odd prime p let $(\frac{a}{p})$ be the Legendre symbol and $q_p(a) = (a^{p-1} - 1)/p$, and let $\mathbb{Z}_p$ be the set of rational numbers whose denominator is not divisible by p. For positive integers a, b and n, if $n = ax^2 + by^2$ for some integers x and y, we briefly write that $n = ax^2 + by^2$.

In Section 2 we find new expressions for T_n, and show that for any prime $p \ne 2, 3, 7$,

$$T_{p-1} \equiv 16^{p-1} \pmod{p^3}, \quad \sum_{n=0}^{p-1}(7n+4)T_n \equiv 4p \pmod{p^2},$$

$$\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{(-4)^n} \equiv (-1)^{\frac{p-1}{2}} p \pmod{p^3}, \quad \sum_{n=0}^{p-1}(2n+1)\frac{T_n}{4^n} \equiv p \pmod{p^4},$$

$$\sum_{n=0}^{p-1} T_n \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 2, 4 \pmod 7 \text{ and so } p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3, 5, 6 \pmod 7 \end{cases}$$

and

(1.3)

$$\sum_{n=0}^{p-1}\frac{T_n}{4^n}\equiv\begin{cases}4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} & \text{if } 4\mid p-1 \text{ and so } p=x^2+y^2 \text{ with } 2\nmid x,\\ -\frac{p^2}{4}\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2}\pmod{p^3} & \text{if } 4\mid p-3.\end{cases}$$

In addition, we pose challenging conjectures for

$$\sum_{n=0}^{p-1}T_n\pmod{p^3},\ \sum_{n=0}^{p-1}\frac{T_n}{16^n}\pmod{p^3},\ \sum_{n=0}^{p-1}\frac{T_n}{(-4)^n}\pmod{p^3},\ \sum_{n=0}^{p-1}(2n+1)\frac{T_n}{(-4)^n}\pmod{p^4},$$

$$\sum_{n=0}^{p-1}(7n+4)T_n\pmod{p^5},\ \sum_{n=0}^{p-1}(7n+3)\frac{T_n}{16^n}\pmod{p^5},\ \sum_{n=0}^{p-1}(2n+1)\frac{T_n}{4^n}\pmod{p^5}$$

and some conjectures on congruences involving $\{A_n\}$, $\{D_n\}$ and $\{b_n\}$.

In Section 3 we prove that for any odd prime p and $x\in\mathbb{Z}_p$ with $(1-8x+32x^2)(1-32x^2)\not\equiv 0\pmod p$,

$$\Big(\sum_{k=0}^{p-1}S_kx^k\Big)^2\equiv\sum_{k=0}^{p-1}\binom{2k}{k}S_k\Big(\frac{x(1-4x)(1-8x)}{(1-32x^2)^2}\Big)^k\pmod p,\tag{1.4}$$

which is the p-analogue of the following identity in [5]:

$$\Big(\sum_{k=0}^{\infty}S_kx^k\Big)^2=\frac{1}{1-32x^2}\sum_{k=0}^{\infty}\binom{2k}{k}S_k\Big(\frac{x(1-4x)(1-8x)}{(1-32x^2)^2}\Big)^k.\tag{1.5}$$

For any prime $p>3$ we also show that for $x\in\mathbb{Z}_p$ with $x^2+1\not\equiv 0\pmod p$,

$$\sum_{k=0}^{p-1}S_k\Big(\frac{x+1}{8}\Big)^k\equiv-\Big(\frac{p}{3}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-3(x^4-x^2+1)n+(x^2+1)(x^2-2)(2x^2-1)}{p}\Big)\pmod p.$$

In addition, for any prime $p>3$ we show that

$$S_{p-1}\equiv(-1)^{\frac{p-1}{2}}(1+5(2^{p-1}-1))\pmod{p^2}\quad\text{and}\quad\sum_{n=1}^{p-1}\frac{nS_n}{4^n}\equiv-1\pmod{p^2}.$$

2. Congruences for sums involving $\{T_n\}$

Recall that

$$T_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}^2\quad(n=0,1,2,\ldots)$$

and

$$(n+1)^3T_{n+1}=(2n+1)(12n(n+1)+4)T_n-16n^3T_{n-1}\quad(n\ge 1).$$

The first few values of T_n are shown below:

$$T_0=1,\ T_1=4,\ T_2=40,\ T_3=544,\ T_4=8536,\ T_5=145504,\ T_6=2618176.$$

Theorem 2.1. *For $n = 0, 1, 2, \ldots$ we have*

$$T_n = \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+2k}{4k} 4^{n-2k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-4)^{n-k} S_k.$$

Proof. Set $T_n' = \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+2k}{4k} 4^{n-2k}$. Then $T_0' = 1 = T_0$ and $T_1' = 4 = T_1$. Using sumtools or Zeilberger's algorithm in Maple (see [17]) we find that $(n+1)^3 T_{n+1}' = (2n+1)(12n(n+1)+4)T_n' - 16n^3 T_{n-1}'$ $(n \ge 1)$. Thus $T_n = T_n'$ as claimed. By [25, Theorem 2.1],

$$\sum_{k=0}^{n} \binom{n}{k} \frac{S_k}{(-4)^k} = \begin{cases} 0 & \text{if } 2 \nmid n, \\ \frac{1}{4^n} \binom{n}{n/2}^2 & \text{if } 2 \mid n. \end{cases}$$

By [26, Theorem 2.2], for any sequence $\{a_n\}$, $\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \Big(a_k - (-1)^{n-k} \sum_{r=0}^{k} \binom{k}{r} a_r \Big) = 0$. Now setting $a_n = \frac{S_n}{(-4)^n}$ we see that

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{S_k}{(-4)^k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \sum_{r=0}^{k} \binom{k}{r} \frac{S_r}{(-4)^r} \\ &= \sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{n+2k}{2k} (-1)^{n-2k} \sum_{r=0}^{2k} \binom{2k}{r} \frac{S_r}{(-4)^r} \\ &= (-1)^n \sum_{k=0}^{[n/2]} \binom{n}{2k} \binom{n+2k}{2k} \frac{1}{4^{2k}} \binom{2k}{k}^2 = \frac{1}{(-4)^n} \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+2k}{4k} \cdot 4^{n-2k} \\ &= \frac{1}{(-4)^n} T_n. \end{aligned}$$

This completes the proof.

Corollary 2.1. *For $|x| < \frac{1}{4}$ we have*

$$\sum_{n=0}^{\infty} T_n x^n = \frac{1}{1-4x} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{x}{(1-4x)^2} \Big)^{2k}.$$

Proof. By Theorem 2.1 and Newton's binomial theorem,

$$\begin{aligned} \sum_{n=0}^{\infty} T_n x^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+2k}{4k} 4^{n-2k} x^n \\ &= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} x^{2k} \sum_{n=2k}^{\infty} \binom{n+2k}{4k} (4x)^{n-2k} \\ &= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} x^{2k} \sum_{r=0}^{\infty} \binom{4k+r}{r} (4x)^r \end{aligned}$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} x^{2k} \sum_{r=0}^{\infty} \binom{-4k-1}{r} (-4x)^r$$
$$= \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} x^{2k} (1-4x)^{-4k-1}.$$

This is the result.

Theorem 2.2. *Let p be an odd prime, $x \in \mathbb{Z}_p$ and $(4x-1)(4x+1) \not\equiv 0 \pmod p$. Then*

$$\sum_{n=0}^{p-1} T_n x^n \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{x}{(1+4x)^2}\Big)^k S_k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{x}{(1-4x)^2}\Big)^{2k} \pmod p.$$

Proof. By Theorem 2.1, $\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \frac{S_k}{(-4)^k} = \frac{T_n}{(-4)^n}$. Thus applying [25, Lemma 2.4] gives

$$\sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{u}{(1-u)^2}\Big)^k \frac{S_k}{(-4)^k} \equiv \sum_{n=0}^{p-1} \frac{T_n}{(-4)^n} u^n \pmod p \quad \text{for } u \not\equiv 1 \pmod p.$$

Replacing u with $-4x$ yields

$$\sum_{n=0}^{p-1} T_n x^n \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{x}{(1+4x)^2}\Big)^k S_k \pmod p.$$

From [25, Theorem 2.6] we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \Big(\frac{x}{(1+4x)^2}\Big)^k S_k \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{4k}{2k} \Big(\frac{x}{(4x-1)^2}\Big)^{2k} \pmod p.$$

Thus the theorem is proved.

Corollary 2.2. *Let p be an odd prime. Then*

$$\sum_{n=0}^{p-1} \frac{T_n}{(-4)^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1,3 \pmod 8 \text{ and so } p = x^2+2y^2, \\ 0 \pmod p & \text{if } p \equiv 5,7 \pmod 8, \end{cases}$$
$$\sum_{n=0}^{p-1} \frac{T_n}{16^n} \equiv \begin{cases} 4x^2 \pmod p & \text{if } p \equiv 1,2,4 \pmod 7 \text{ and so } p = x^2+7y^2, \\ 0 \pmod p & \text{if } p \equiv 3,5,6 \pmod 7. \end{cases}$$

Proof. It is easy to verify the results for $p = 3, 7$. Now assume $p \neq 3, 7$. By Theorem 2.1,

$$\sum_{n=0}^{p-1} \frac{T_n}{(-4)^n} = \sum_{n=0}^{p-1} \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+2k}{4k} (-1)^n 4^{-2k}$$
$$= \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{4k}{2k} \frac{1}{16^k} \sum_{n=2k}^{p-1} \binom{n+2k}{4k} (-1)^{n-2k}.$$

Since

$$\sum_{n=2k}^{p-1}\binom{n+2k}{4k}(-1)^{n-2k}=\sum_{r=0}^{p-1-2k}\binom{4k+r}{r}(-1)^r=\sum_{r=0}^{p-1-2k}\binom{-4k-1}{r}$$
$$\equiv\sum_{r=0}^{p-1-2k}\binom{p-1-4k}{r}=2^{p-1-4k}\equiv\frac{1}{16^k}\pmod p,$$

combining the above with [14, Theorem 4(3)] gives the first result. Taking $x=\frac{1}{16}$ in Theorem 2.2 gives

$$\sum_{n=0}^{p-1}\frac{T_n}{16^n}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k}\pmod p.$$

Now applying [23, Theorem 5.2] yields the second result.

Theorem 2.3. *Let p be an odd prime. Then $T_{p-1}\equiv 16^{p-1}\pmod{p^3}$.*

Proof. Clearly the result holds for $p=3$. Now assume $p>3$. By Theorem 2.1,

$$\begin{aligned}T_{p-1}&=\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\binom{4k}{2k}\binom{p-1+2k}{4k}4^{p-1-2k}\\&=4^{p-1}+\sum_{k=1}^{(p-1)/2}\frac{(p+2k-1)(p+2k-2)\cdots(p+1)p(p-1)\cdots(p-2k)}{k!^4}\cdot 4^{p-1-2k}\\&=4^{p-1}+p\sum_{k=1}^{(p-1)/2}\frac{(p^2-1^2)\cdots(p^2-(2k)^2)}{(2k+p)\cdot k!^4}\cdot 4^{p-1-2k}\\&\equiv 4^{p-1}+p\sum_{k=1}^{(p-1)/2}\frac{(2k-p)\cdot(2k)!^2}{(4k^2-p^2)\cdot k!^4}\cdot 4^{p-1-2k}\equiv 4^{p-1}+4^{p-1}p\sum_{k=1}^{(p-1)/2}\frac{2k-p}{4k^2}\cdot\frac{\binom{2k}{k}^2}{16^k}\\&=4^{p-1}+4^{p-1}p\Big(\frac12\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^kk}-\frac p4\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^kk^2}\Big)\pmod{p^3}.\end{aligned}$$

By [33, Theorem 4],

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^kk^2}\equiv-\frac12\Big(\frac{1-16^{-(p-1)}}{p}\Big)^2\pmod p,$$
$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^kk}\equiv\frac{1-16^{-(p-1)}}{p}+\frac p2\Big(\frac{1-16^{-(p-1)}}{p}\Big)^2\pmod{p^2}.$$

Clearly,

$$\begin{aligned}\frac{1-16^{-(p-1)}}{p}&=\frac{(2^{p-1}-1+1)^4-1}{16^{p-1}p}\equiv\frac{6(2^{p-1}-1)^2+4(2^{p-1}-1)}{16^{p-1}p}\\&\equiv\frac{4q_p(2)+6pq_p(2)^2}{(1+pq_p(2))^4}\equiv\frac{4q_p(2)+6pq_p(2)^2}{1+4pq_p(2)}\\&\equiv(1-4pq_p(2))(4q_p(2)+6pq_p(2)^2)\equiv 4q_p(2)-10pq_p(2)^2\pmod{p^2}.\end{aligned}$$

Thus,

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k k^2}\equiv -8q_p(2)^2 \pmod p,$$
$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k k}\equiv 4q_p(2)-10pq_p(2)^2+8pq_p(2)^2=4q_p(2)-2pq_p(2)^2 \pmod{p^2}.$$

Therefore,

$$\begin{aligned}T_{p-1}&\equiv 4^{p-1}+4^{p-1}p\Big(\frac12\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k k}-\frac p4\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k k^2}\Big)\\&\equiv 4^{p-1}+4^{p-1}p\Big(2q_p(2)-pq_p(2)^2-\frac p4(-8q_p(2)^2)\Big) \pmod{p^3}\end{aligned}$$

Since $4^{p-1}=(1+pq_p(2))^2=1+2pq_p(2)+p^2q_p(2)^2$ we deduce that

$$\begin{aligned}T_{p-1}&\equiv 1+2pq_p(2)+p^2q_p(2)^2+p(1+2pq_p(2))(2q_p(2)+pq_p(2)^2)\\&\equiv 1+4pq_p(2)+6p^2q_p(2)^2\equiv (1+pq_p(2))^4=16^{p-1} \pmod{p^3}.\end{aligned}$$

This proves the theorem.

Theorem 2.4. *Let p be a prime with $p\ne 2,7$. Then*

$$\sum_{n=0}^{p-1}T_n\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and so } p=x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. Since $p\mid\binom{2k}{k}$ for $\frac p2<k<p$ and $T_n=\sum_{k=0}^n\binom nk^2\binom{2k}{n}^2=\sum_{k=0}^n\binom{2k}{k}^2\binom{k}{n-k}^2$, we see that

$$\begin{aligned}\sum_{n=0}^{p-1}T_n&=\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{k}{n-k}^2=\sum_{k=0}^{p-1}\binom{2k}{k}^2\sum_{n=k}^{p-1}\binom{k}{n-k}^2\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2\sum_{r=0}^{p-1-k}\binom kr^2\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\sum_{r=0}^{k}\binom kr^2=\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^3 \pmod{p^2}.\end{aligned}$$

Now applying [23, Theorems 3.3 and 3.4] yields the result.

Theorem 2.5. *Let p be a prime with $p\ne 2,3,7$. Then*

$$\sum_{n=0}^{p-1}(7n+4)T_n\equiv 4p \pmod{p^2}.$$

Proof. Since $p\mid\binom{2k}{k}$ for $\frac p2<k<p$ we see that

$$\begin{aligned}\sum_{n=0}^{p-1}(7n+4)T_n&=\sum_{n=0}^{p-1}(7n+4)\sum_{k=0}^{n}\binom{2k}{k}^2\binom{k}{n-k}^2\\&=\sum_{k=0}^{p-1}\binom{2k}{k}^2\sum_{n=k}^{p-1}(7n+4)\binom{k}{n-k}^2\end{aligned}$$

$$= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \sum_{r=0}^{p-1-k} (7k+4+7r) \binom{k}{r}^2$$

$$\equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \sum_{r=0}^{k} (7k+4+7r) \binom{k}{r}^2 \pmod{p^2}.$$

It is well known that (see [9, (3.77)-(3.78)]) $\sum_{r=0}^{k} \binom{k}{r}^2 = \binom{2k}{k}$ and $\sum_{r=0}^{k} r\binom{k}{r}^2 = \frac{k}{2}\binom{2k}{k}$. Thus,

$$\sum_{r=0}^{k} (7k+4+7r) \binom{k}{r}^2 = (7k+4)\binom{2k}{k} + \frac{7k}{2}\binom{2k}{k} = \frac{21k+8}{2}\binom{2k}{k}.$$

Hence

$$\sum_{n=0}^{p-1} (7n+4) T_n$$

$$\equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \sum_{r=0}^{k} (7k+4+7r) \binom{k}{r}^2 \equiv \frac{1}{2} \sum_{k=0}^{(p-1)/2} (21k+8) \binom{2k}{k}^3 \pmod{p^2}.$$

By [28, Theorem 1.3], $\sum_{k=0}^{p-1} (21k+8)\binom{2k}{k}^3 \equiv 8p \pmod{p^4}$. Thus the result follows.

Theorem 2.6. *Let $p > 3$ be a prime. Then*

$$\sum_{n=0}^{p-1} (2n+1) \frac{T_n}{4^n} \equiv p \pmod{p^4}.$$

Proof. By Theorem 2.1,

$$\sum_{n=0}^{p-1} (2n+1) \frac{T_n}{4^n} = \sum_{n=0}^{p-1} (2n+1) \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{4k}{2k} \binom{n+2k}{4k} 4^{-2k}$$

$$= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{16^k} \sum_{n=2k}^{p-1} (2n+1) \binom{n+2k}{4k}.$$

Using induction one can easily verify that

$$\sum_{n=2k}^{p-1} (2n+1) \binom{n+2k}{4k} = \frac{p(p-2k)}{2k+1} \binom{p+2k}{4k}$$

$$= \frac{p^2}{2k+1} \cdot \frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)}{(4k)!}.$$

Thus,

$$\sum_{n=0}^{p-1} (2n+1) \frac{T_n}{4^n} = \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{16^k} \cdot \frac{p^2}{2k+1} \cdot \frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)}{(4k)!}$$

$$= \sum_{k=0}^{(p-1)/2} \frac{p^2}{16^k (2k+1)} \cdot \frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)}{k!^4}$$

$$\equiv \sum_{k=0}^{(p-1)/2} \frac{p^2}{16^k(2k+1)} \cdot \frac{(-1^2)(-2^2)\cdots(-(2k)^2)(1-p^2\sum_{i=1}^{2k}\frac{1}{i^2})}{k!^4}$$
$$= \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2 p^2}{16^k(2k+1)}\Big(1-p^2\sum_{i=1}^{2k}\frac{1}{i^2}\Big) \pmod{p^5}.$$

It is known that (see [12] or [19, Corollary 5.1]) $\sum_{i=1}^{p-1}\frac{1}{i^2} \equiv 0 \pmod p$. Thus,

$$\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{4^n} \equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{\binom{2k}{k}^2 p^2}{16^k(2k+1)} = 16^{-\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}}^2 p + p^2\sum_{k=0}^{\frac{p-3}{2}} \frac{\binom{2k}{k}^2}{16^k(2k+1)} \pmod{p^4}.$$

By Morley's congruence (see [31, Lemma 2.2]) , $\binom{p-1}{(p-1)/2} \equiv (-1)^{\frac{p-1}{2}}4^{p-1} \pmod{p^3}$. From [31, Theorem 1.2] we know that $\sum_{k=0}^{\frac{p-3}{2}}\frac{\binom{2k}{k}^2}{16^k(2k+1)} \equiv -2q_p(2)-pq_p(2)^2 \pmod{p^2}$. Thus,

$$\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{4^n} \equiv 4^{p-1}p - 2q_p(2)p^2 - q_p(2)^2p^3 = p \pmod{p^4}.$$

This proves the theorem.

Theorem 2.7. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{(-4)^n} \equiv (-1)^{\frac{p-1}{2}}p \pmod{p^3}.$$

Proof. By Theorem 2.1,

$$\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{(-4)^n} = \sum_{n=0}^{p-1}(2n+1)(-1)^n\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{4k}{2k}\binom{n+2k}{4k}4^{-2k}$$
$$= \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{16^k}\sum_{n=2k}^{p-1}(2n+1)(-1)^n\binom{n+2k}{4k}.$$

By [29, (3.5)], $\sum_{n=2k}^{p-1}(2n+1)(-1)^n\binom{n+2k}{4k} = (p-2k)\binom{p+2k}{4k}$. Thus,

$$\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{(-4)^n}$$
$$= \sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{16^k}(p-2k)\binom{p+2k}{4k} = \sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{16^k}\cdot\frac{p(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)}{(4k)!}$$
$$= p\sum_{k=0}^{(p-1)/2}\frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)}{16^k\cdot k!^4} \equiv p\sum_{k=0}^{(p-1)/2}\frac{(2k)!^2(1-p^2\sum_{i=1}^{2k}\frac{1}{i^2})}{16^k\cdot k!^4}$$
$$= p\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\Big(1-p^2\sum_{i=1}^{2k}\frac{1}{i^2}\Big) \pmod{p^5}.$$

From [14] or [21] we know that $\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{\frac{p-1}{2}}\pmod{p^2}$. Thus the result follows.

The Bernoulli numbers $\{B_n\}$, Euler numbers $\{E_n\}$ and the sequence $\{U_n\}$ are defined by

$$\begin{aligned}&B_0=1,\quad \sum_{k=0}^{n-1}\binom{n}{k}B_k=0\quad(n\ge 2),\\&E_0=1,\quad E_n=-\sum_{k=1}^{[n/2]}\binom{n}{2k}E_{n-2k}\quad(n\ge 1),\\&U_0=1,\quad U_n=-2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k}\quad(n\ge 1).\end{aligned}$$

For congruences involving B_n,E_n and U_n see [19,20,22].

Theorem 2.8. *Suppose that p is an odd prime. Then*

$$\sum_{n=0}^{p-1}\frac{T_n}{4^n}\equiv\begin{cases}\frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1-\frac{p^2}{2}E_{p-3}\Big)\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}\\ \qquad\qquad\text{if } 4\mid p-1 \text{ and so } p=x^2+y^2 \text{ with } 2\nmid x,\\ -\frac{p^2}{4}\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2}\pmod{p^3}\quad \text{if } 4\mid p-3.\end{cases}$$

Proof. By Theorem 2.1 and [9, (1.49)],

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n}{4^n}&=\sum_{n=0}^{p-1}\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{4k}{2k}\binom{n+2k}{4k}4^{-2k}\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{16^k}\sum_{n=2k}^{p-1}\binom{n+2k}{4k}=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{16^k}\binom{p+2k}{4k+1}\\&=\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{16^k}\cdot\frac{p}{4k+1}\cdot\frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)}{(4k)!}\\&=\sum_{k=0}^{(p-1)/2}\frac{p}{16^k(4k+1)}\cdot\frac{(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)}{k!^4}.\end{aligned}$$

Obviously,

$$(p^2-1^2)(p^2-2^2)\cdots(p^2-(2k)^2)\equiv(2k)!^2\Big(1-p^2\sum_{i=1}^{2k}\frac{1}{i^2}\Big)\pmod{p^4}.$$

Thus,

$$\sum_{n=0}^{p-1}\frac{T_n}{4^n}\equiv\sum_{k=0}^{(p-1)/2}\frac{p}{16^k(4k+1)}\binom{2k}{k}^2\Big(1-p^2\sum_{i=1}^{2k}\frac{1}{i^2}\Big)\pmod{p^4}.$$

If $k=\frac{p-1}{4}$, then $4k+1=p$ and $\sum_{i=1}^{2k}\frac{1}{i^2}=\sum_{i=1}^{(p-1)/2}\frac{1}{i^2}\equiv 0\pmod p$ by [19, Theorem 5.2]. If $k\in\{0,1,\ldots,\frac{p-1}{2}\}$ and $k\neq\frac{p-1}{4}$, then $p\nmid 4k+1$. Thus,

$$\sum_{n=0}^{p-1}\frac{T_n}{4^n}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2 p}{16^k(4k+1)}\pmod{p^3}. \tag{2.1}$$

By [21, Lemma 2.2], for $k=1,2,\ldots,\frac{p-1}{2}$,

$$\binom{\frac{p-1}{2}+k}{2k}\equiv\frac{\binom{2k}{k}}{(-16)^k}\Big(1-p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\Big)\pmod{p^4}.$$

Thus,

$$\frac{\binom{2k}{k}}{16^k}\equiv(-1)^k\binom{\frac{p-1}{2}+k}{2k}\Big(1+p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\Big)\pmod{p^4}.$$

Therefore,

$$\begin{aligned}
&\sum_{n=0}^{p-1}\frac{T_n}{4^n}\\
&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2 p}{16^k(4k+1)}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}(-1)^k\binom{\frac{p-1}{2}+k}{2k}\frac{p}{4k+1}\Big(1+p^2\sum_{i=1}^{k}\frac{1}{(2i-1)^2}\Big)\\
&\equiv\begin{cases}\displaystyle\sum_{k=0}^{(p-1)/2}\binom{2k}{k}(-1)^k\binom{\frac{p-1}{2}+k}{2k}\frac{p}{4k+1}\pmod{p^3} & \text{if } p\equiv 3\pmod 4,\\[2ex]
\displaystyle\sum_{k=0}^{(p-1)/2}\binom{2k}{k}(-1)^k\binom{\frac{p-1}{2}+k}{2k}\frac{p}{4k+1} & \\
\displaystyle\quad+p^2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}(-1)^{\frac{p-1}{4}}\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}}\sum_{i=1}^{(p-1)/4}\frac{1}{(2i-1)^2}\pmod{p^3} & \text{if } p\equiv 1\pmod 4.\end{cases}
\end{aligned}$$

From [9, (3.100)] we know that

$$\sum_{k=0}^{n}\binom{2k}{k}(-1)^k\binom{n+k}{2k}\frac{x+n}{x+k}=(-1)^n\frac{(x-1)(x-2)\cdots(x-n)}{x(x+1)(x+2)\cdots(x+n-1)}. \tag{2.2}$$

Thus,

$$\begin{aligned}
&\sum_{k=0}^{(p-1)/2}\binom{2k}{k}(-1)^k\binom{\frac{p-1}{2}+k}{2k}\frac{\frac14+\frac{p-1}{2}}{\frac14+k}\\
&=(-1)^{\frac{p-1}{2}}\frac{(\frac14-1)(\frac14-2)\cdots(\frac14-\frac{p-1}{2})}{\frac14(\frac14+1)\cdots(\frac14+\frac{p-1}{2}-1)}=\frac{3\cdot 7\cdots(2p-3)}{1\cdot 5\cdots(2p-5)}\\
&=\begin{cases}\dfrac{(p^2-2^2)(p^2-6^2)\cdots(p^2-(p-3)^2)}{p(p^2-4^2)(p^2-8^2)\cdots(p^2-(p-5)^2)} & \text{if } p\equiv 1\pmod 4,\\[2ex]
\dfrac{p(p^2-4^2)(p^2-8^2)\cdots(p^2-(p-3)^2)}{(p^2-2^2)(p^2-6^2)\cdots(p^2-(p-5)^2)} & \text{if } p\equiv 3\pmod 4.\end{cases}
\end{aligned}$$

If $p\equiv 3\pmod 4$, from the above we see that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n}{4^n}&\equiv\frac{p}{2p-1}\sum_{k=0}^{(p-1)/2}\binom{2k}{k}(-1)^k\binom{\frac{p-1}{2}+k}{2k}\frac{2p-1}{4k+1}\\&\equiv\frac{p^2}{2p-1}\cdot\frac{(-4^2)(-8^2)\cdots(-(p-3)^2)}{(-2^2)(-6^2)\cdots(-(p-5)^2)}\\&=\frac{p^2}{2p-1}\cdot\frac{(4\cdot 8\cdots(p-3))^4}{(2\cdot 4\cdots(p-3))^2}=\frac{p^2}{2p-1}\cdot\frac{(4^{\frac{p-3}{4}}\cdot(\frac{p-3}{4})!)^4}{(2^{\frac{p-3}{2}}\cdot(\frac{p-3}{2})!)^2}\\&=\frac{p^2}{2p-1}\cdot 2^{p-3}\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2}\equiv-\frac{p^2}{4}\binom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2}\pmod{p^3}.\end{aligned}$$

If $p\equiv 1\pmod 4$, from the above we see that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n}{4^n}&\equiv\frac{1}{2p-1}\sum_{k=0}^{(p-1)/2}\binom{2k}{k}(-1)^k\binom{\frac{p-1}{2}+k}{2k}\frac{(2p-1)p}{4k+1}\\&\quad+p^2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}(-1)^{\frac{p-1}{4}}\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}}\sum_{i=1}^{(p-1)/4}\frac{1}{(2i-1)^2}\\&=\frac{1}{2p-1}\cdot\frac{(p^2-2^2)(p^2-6^2)\cdots(p^2-(p-3)^2)}{(p^2-4^2)(p^2-8^2)\cdots(p^2-(p-5)^2)}\\&\quad+p^2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}(-1)^{\frac{p-1}{4}}\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}}\sum_{k=1}^{(p-1)/4}\frac{1}{(2k-1)^2}\\&\equiv\frac{1}{2p-1}\cdot\frac{(-2^2)(-6^2)\cdots(-(p-3)^2)(1-p^2\sum_{k=1}^{(p-1)/4}\frac{1}{(4k-2)^2})}{(-4^2)(-8^2)\cdots(-(p-5)^2)(1-p^2\sum_{k=1}^{(p-5)/4}\frac{1}{(4k)^2})}\\&\quad+p^2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}(-1)^{\frac{p-1}{4}}\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}}\sum_{k=1}^{(p-1)/4}\frac{1}{(2k-1)^2}\pmod{p^3}.\end{aligned}$$

By [12] or [20], $\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv 0\pmod p$ and $\sum_{k=1}^{(p-1)/4}\frac{1}{k^2}\equiv 4E_{p-3}\pmod p$. Thus,

$$\sum_{k=1}^{(p-1)/4}\frac{1}{(2k-1)^2}=\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}-\sum_{k=1}^{(p-1)/4}\frac{1}{(2k)^2}\equiv -E_{p-3}\pmod p.$$

Now, from the above we deduce that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n}{4^n}&\equiv\frac{1}{1-2p}\cdot\frac{2^2\cdot 6^2\cdots(p-3)^2\cdot(1+\frac{p^2}{4}E_{p-3})}{4^2\cdot 8^2\cdots(p-5)^2\cdot(1+p^2(1-\frac14E_{p-3}))}\\&\quad-p^2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}(-1)^{\frac{p-1}{4}}\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}}E_{p-3}\pmod{p^3}.\end{aligned}$$

It is clear that

$$\frac{2^2\cdot 6^2\cdots(p-3)^2}{4^2\cdot 8^2\cdots(p-5)^2}=\frac{2^2\cdot 4^2\cdots(p-3)^2}{(4\cdot 8\cdots(p-5))^4}=\frac{2^{p-3}\cdot(\frac{p-3}{2})!^2}{4^{p-5}\cdot(\frac{p-5}{4})!^4}$$

$$= \frac{2^{p-3}}{2^{2p-10}} \cdot \frac{(p-1)^2}{64} \cdot \frac{(\frac{p-1}{2})!^2}{(\frac{p-1}{4})!^4} = \frac{(p-1)^2}{2^{p-1}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2$$

and

$$\frac{1+\frac{p^2}{4}E_{p-3}}{1+p^2(1-\frac{1}{4}E_{p-3})} \equiv \Big(1+\frac{p^2}{4}E_{p-3}\Big)\Big(1-p^2\Big(1-\frac{1}{4}E_{p-3}\Big)\Big) \equiv 1-p^2\Big(1-\frac{1}{2}E_{p-3}\Big) \pmod{p^3}.$$

By [21, Lemma 2.5], $\binom{\frac{3(p-1)}{4}}{\frac{p-1}{2}} = \binom{\frac{p-1}{2}+\frac{p-1}{4}}{\frac{p-1}{4}} \equiv (-1)^{\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p$. Thus,

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n}{4^n} &\equiv \frac{1}{1-2p}\cdot\frac{(p-1)^2}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1-p^2\Big(1-\frac{1}{2}E_{p-3}\Big)\Big) - p^2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 E_{p-3} \\ &\equiv \frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2(1+p^2)\Big(1-p^2\Big(1-\frac{1}{2}E_{p-3}\Big)\Big) - p^2\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 E_{p-3} \\ &\equiv \frac{1}{2^{p-1}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\Big(1-\frac{p^2}{2}E_{p-3}\Big) \pmod{p^3}.\end{aligned}$$

Suppose $p = x^2+y^2$ with $x \equiv 1 \pmod 4$. By [7, Theorem 3],

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv \Big(2x-\frac{p}{2x}-\frac{p^2}{8x^3}\Big)\Big(1+\frac{1}{2}pq_p(2)+\frac{p^2}{8}(2E_{p-3}-q_p(2)^2)\Big) \pmod{p^3}.$$

This yields

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2x + p\Big(xq_p(2)-\frac{1}{2x}\Big) + p^2\Big(\frac{x}{4}(2E_{p-3}-q_p(2)^2) - \frac{1}{4x}q_p(2) - \frac{1}{8x^3}\Big) \pmod{p^3}.$$

Hence

$$\begin{aligned}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2 &\equiv \Big(2x+p\Big(xq_p(2)-\frac{1}{2x}\Big)\Big)^2 + 4xp^2\Big(\frac{x}{4}(2E_{p-3}-q_p(2)^2) - \frac{1}{4x}q_p(2) - \frac{1}{8x^3}\Big) \\ &\equiv 4x^2 + p(4x^2q_p(2)-2) + p^2\Big(-2q_p(2)+2x^2E_{p-3}-\frac{1}{4x^2}\Big) \pmod{p^3}.\end{aligned}$$

Since

$$\frac{1}{2^{p-1}} = \frac{1}{1+pq_p(2)} = \frac{1-pq_p(2)+p^2q_p(2)^2}{1+p^3q_p(2)^3} \equiv 1-pq_p(2)+p^2q_p(2)^2 \pmod{p^3},$$

we see that

$$\begin{aligned}\frac{1}{2^{p-1}}\Big(1-\frac{p^2}{2}E_{p-3}\Big) &\equiv (1-pq_p(2)+p^2q_p(2)^2)\Big(1-\frac{p^2}{2}E_{p-3}\Big) \\ &\equiv 1-pq_p(2)+p^2\Big(q_p(2)^2-\frac{1}{2}E_{p-3}\Big) \pmod{p^3}.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{p-1}\frac{T_n}{4^n} \equiv \binom{\frac{p-1}{2}}{\frac{p-1}{4}}^2\frac{1}{2^{p-1}}\Big(1-\frac{p^2}{2}E_{p-3}\Big)$$

$$\equiv \Big(4x^2+p(4x^2q_p(2)-2)+p^2\Big(-2q_p(2)+2x^2E_{p-3}-\frac{1}{4x^2}\Big)\Big)$$
$$\times\Big(1-pq_p(2)+p^2\Big(q_p(2)^2-\frac12E_{p-3}\Big)\Big)$$
$$\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

This proves the result in the case $p\equiv 1\pmod 4$. The proof is now complete.

Based on calculations with Maple, we pose the following challenging conjectures:

Conjecture 2.1. *Let $p>3$ be a prime and $n,r\in\mathbb{Z}^+$. Then*

$$A_{np^r-1}\equiv A_{np^{r-1}-1}+p^{3r}\cdot\frac{A_{np-1}-A_{n-1}}{p^3}\pmod{p^{3r+1}},$$
$$D_{np^r-1}\equiv 64^{np^{r-1}(p-1)}D_{np^{r-1}-1}+p^{3r}\cdot\frac{D_{np-1}-64^{n(p-1)}D_{n-1}}{p^3}\pmod{p^{3r+1}},$$
$$b_{np^r-1}\equiv 81^{np^{r-1}(p-1)}b_{np^{r-1}-1}+p^{3r}\cdot\frac{b_{np-1}-81^{n(p-1)}b_{n-1}}{p^3}\pmod{p^{3r+1}},$$
$$T_{np^r-1}\equiv 16^{np^{r-1}(p-1)}T_{np^{r-1}-1}+p^{3r}\cdot\frac{T_{np-1}-16^{n(p-1)}T_{n-1}}{p^3}\pmod{p^{3r+1}}.$$

Remark 2.1 Let $p>3$ be a prime and $n,r\in\mathbb{Z}^+$. In 1985 Beukers [2] proved that $A_{np^r-1}\equiv A_{np^{r-1}-1}\pmod{p^{3r}}$.

Conjecture 2.2. *Let p be a prime with $p\neq 2,7$.*

(i) *If $p\equiv 1,2,4\pmod 7$ and so $p=x^2+7y^2$, then*

$$\sum_{n=0}^{p-1}T_n\equiv\sum_{n=0}^{p-1}\frac{T_n}{16^n}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If $p\equiv 3,5,6\pmod 7$, then*

$$\sum_{n=0}^{p-1}T_n\equiv-\frac{20}{29}\sum_{n=0}^{p-1}\frac{T_n}{16^n}\equiv\begin{cases}\frac{5}{16}p^2\binom{3[p/7]}{[p/7]}^{-2}\pmod{p^3} & \text{if } p\equiv 3\pmod 7,\\ \frac{45}{256}p^2\binom{3[p/7]}{[p/7]}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 7,\\ \frac{125}{7744}p^2\binom{3[p/7]}{[p/7]}^{-2}\pmod{p^3} & \text{if } p\equiv 6\pmod 7.\end{cases}$$

Conjecture 2.3. *Let p be an odd prime.*

(i) *If $p\equiv 1,3\pmod 8$ and so $p=x^2+2y^2$, then*

$$\sum_{n=0}^{p-1}\frac{T_n}{(-4)^n}\equiv(-1)^{\frac{p-1}{2}}\sum_{n=0}^{p-1}\binom{2n}{n}\frac{S_n}{32^n}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

(ii) *If $p\equiv 5,7\pmod 8$ and $p\neq 5,7$, then*

$$\sum_{n=0}^{p-1}\frac{T_n}{(-4)^n}\equiv\frac{15}{7}(-1)^{\frac{p-1}{2}}\sum_{n=0}^{p-1}\binom{2n}{n}\frac{S_n}{32^n}\equiv\begin{cases}-\frac{5}{9}p^2\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } 8\mid p-5,\\ \frac{5}{2}p^2\binom{[p/4]}{[p/8]}^{-2}\pmod{p^3} & \text{if } 8\mid p-7.\end{cases}$$

Conjecture 2.4. *Let $p>3$ be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1}(7n+4)T_n &\equiv 4p+\frac{25}{3}p^4B_{p-3} \pmod{p^5},\\
\sum_{n=0}^{p-1}(7n+3)\frac{T_n}{16^n} &\equiv 3p+\frac{25}{12}p^4B_{p-3} \pmod{p^5},\\
\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{4^n} &\equiv p+\frac{7}{6}p^4B_{p-3} \pmod{p^5},\\
\sum_{n=0}^{p-1}(2n+1)\frac{T_n}{(-4)^n} &\equiv (-1)^{\frac{p-1}{2}}p+p^3E_{p-3} \pmod{p^4}.
\end{aligned}$$

Conjecture 2.5. *Let $p>3$ be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1}(4n+3)b_n &\equiv 3p\Big(\frac{p}{3}\Big)+21p^3U_{p-3} \pmod{p^4},\\
\sum_{n=0}^{p-1}(4n+3)\frac{b_n}{(-3)^n} &\equiv 3p\Big(\frac{p}{3}\Big)+14p^3U_{p-3} \pmod{p^4},\\
\sum_{n=0}^{p-1}(4n+1)\frac{b_n}{(-27)^n} &\equiv p\Big(\frac{p}{3}\Big)-2p^3U_{p-3} \pmod{p^4},\\
\sum_{n=0}^{p-1}(4n+1)\frac{b_n}{81^n} \equiv \sum_{n=0}^{p-1}(2n+1)\frac{b_n}{(-9)^n} &\equiv p\Big(\frac{p}{3}\Big)+p^3U_{p-3} \pmod{p^4},\\
\sum_{n=0}^{p-1}(2n+1)\frac{b_n}{9^n} &\equiv p\Big(\frac{p}{3}\Big)+\frac{5}{2}p^3U_{p-3} \pmod{p^4}.
\end{aligned}$$

Conjecture 2.6. *Let $p>3$ be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1}(5n+4)D_n &\equiv 4p\Big(\frac{p}{3}\Big)+28p^3U_{p-3} \pmod{p^4},\\
\sum_{n=0}^{p-1}(3n+2)\frac{D_n}{(-2)^n} &\equiv 2(-1)^{\frac{p-1}{2}}p+6p^3E_{p-3} \pmod{p^4},\\
\sum_{n=0}^{p-1}(2n+1)\frac{D_n}{(-8)^n} &\equiv p\Big(\frac{p}{3}\Big)+\frac{5}{2}p^3U_{p-3} \pmod{p^4},\\
\sum_{n=0}^{p-1}(2n+1)\frac{D_n}{8^n} &\equiv p+\frac{35}{24}p^4B_{p-3} \pmod{p^5},\\
\sum_{n=0}^{p-1}(5n+1)\frac{D_n}{64^n} &\equiv p\Big(\frac{p}{3}\Big)-2p^3U_{p-3} \pmod{p^4}.
\end{aligned}$$

Remark 2.2 Let p be a prime greater than 3. In [24] the author conjectured the congruences in Conjecture 2.5 modulo p^3. In [30] Z.W. Sun made conjectures for

$$\sum_{n=0}^{p-1}(5n+4)D_n,\quad \sum_{n=0}^{p-1}(3n+2)\frac{D_n}{(-2)^n},\quad \sum_{n=0}^{p-1}(2n+1)\frac{D_n}{(-8)^n}\pmod{p^3}$$

and $\sum_{n=0}^{p-1}(2n+1)\frac{D_n}{8^n}\pmod{p^4}$.

3. Congruences for sums involving S_n

Let $\{P_n(x)\}$ be the famous Legendre polynomials given by

$$P_n(x)=\sum_{k=0}^{n}\binom nk\binom{n+k}k\Big(\frac{x-1}2\Big)^k=\frac1{2^n}\sum_{k=0}^{[n/2]}\binom nk(-1)^k\binom{2n-2k}n x^{n-2k}.$$

It is well known that

$$P_0(x)=1,\ P_1(x)=x,\ (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)\quad(n\ge1).$$

Based on the results in [25], in this section we deduce further congruences for the Apéry-like sequence

$$S_n=\sum_{k=0}^{[n/2]}\binom{2k}k^2\binom n{2k}4^{n-2k}.$$

We first prove the p-analogue of (1.5).

Theorem 3.1. *Let p be an odd prime, $x\in\mathbb{Z}_p$ and $(1-8x+32x^2)(1-32x^2)\not\equiv0\pmod p$. Then*

$$\Big(\sum_{k=0}^{p-1}S_kx^k\Big)^2\equiv\sum_{k=0}^{p-1}\binom{2k}kS_k\Big(\frac{x(1-4x)(1-8x)}{(1-32x^2)^2}\Big)^k\pmod p.$$

Proof. Since $S_0=1$, the result is true for $x\equiv0\pmod p$. From [15] we know that

(3.1)

$$\sum_{k=0}^{p-1}\frac{S_k}{4^k}\equiv1+2(-1)^{\frac{p-1}2}p^2E_{p-3}\pmod{p^3},\quad \sum_{k=0}^{p-1}\frac{S_k}{8^k}\equiv(-1)^{\frac{p-1}2}-p^2E_{p-3}\pmod{p^3}.$$

Thus, the result is true for $x\equiv\frac14,\frac18\pmod p$. Now assume $x(1-4x)(1-8x)\not\equiv0\pmod p$. By [25, Theorem 2.13],

$$\sum_{k=0}^{p-1}S_kx^k\equiv\sum_{k=0}^{\frac{p-1}2}\binom{2k}k^2\Big(4-\frac1x\Big)^{-2k}=\sum_{k=0}^{\frac{p-1}2}\binom{2k}k^2\Big(\frac x{4x-1}\Big)^{2k}\pmod p.$$

Set $t=1-\frac{32x^2}{(4x-1)^2}$. Then $\frac{1-t}{32}=\frac{x^2}{(4x-1)^2}$. By [21, (2.4)], $P_{\frac{p-1}2}(t)\equiv\sum_{k=0}^{\frac{p-1}2}\binom{2k}k^2(\frac{1-t}{32})^k\pmod{p^2}$. Hence

$$\sum_{k=0}^{p-1}S_kx^k\equiv\sum_{k=0}^{\frac{p-1}2}\binom{2k}k^2\Big(\frac{x^2}{(4x-1)^2}\Big)^k\equiv P_{\frac{p-1}2}(t)\pmod p.$$

By [21, Theorem 2.6], $P_{\frac{p-1}{2}}(t) \equiv (\frac{2(t+1)}{p})P_{\frac{p-1}{2}}(\frac{3-t}{1+t}) \pmod p$. Since $\frac{3-t}{1+t} = 1 + \frac{32x^2}{1-8x}$ we see that

$$\sum_{k=0}^{p-1} S_k x^k \equiv P_{\frac{p-1}{2}}(t) \equiv (\frac{2(t+1)}{p})P_{\frac{p-1}{2}}(\frac{3-t}{1+t}) = \Big(\frac{1-8x}{p}\Big)P_{\frac{p-1}{2}}\Big(1+\frac{32x^2}{1-8x}\Big) \pmod p.$$

By [23, Lemma 2.3], for $u \in \mathbb{Z}_p$ with $u \not\equiv 0 \pmod p$,

$$u^{\frac{p-1}{2}} P_{\frac{p-1}{2}}(u) \equiv (-1)^{[\frac{p}{4}]} \sum_{k=0}^{[p/4]} \frac{\binom{2k}{k}\binom{4k}{2k}}{(8u)^{2k}} \pmod p.$$

Hence

$$\begin{aligned}\sum_{k=0}^{p-1} S_k x^k &\equiv \Big(\frac{1-8x}{p}\Big)P_{\frac{p-1}{2}}\Big(\frac{1-8x+32x^2}{1-8x}\Big)\\ &\equiv \Big(\frac{1-8x+32x^2}{p}\Big)(-1)^{[\frac{p}{4}]}\sum_{k=0}^{[p/4]}\binom{2k}{k}\binom{4k}{2k}\Big(\frac{1-8x}{8(1-8x+32x^2)}\Big)^{2k} \pmod p.\end{aligned} \tag{3.2}$$

From [23, Theorem 4.1] we know that

$$\Big(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}u^k\Big)^2 \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}(u(1-64u))^k \pmod{p^2}.$$

Therefore, noting that $p \mid \binom{2k}{k}\binom{4k}{2k}$ for $\frac{p}{4} < k < p$ and then applying the above we obtain

$$\begin{aligned}\Big(\sum_{k=0}^{p-1} S_k x^k\Big)^2 &\equiv \Big(\sum_{k=0}^{p-1}\binom{2k}{k}\binom{4k}{2k}\Big(\frac{1-8x}{8(1-8x+32x^2)}\Big)^{2k}\Big)^2\\ &\equiv \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{(1-8x)^2}{8^2(1-8x+32x^2)^2}\Big(1-\frac{(1-8x)^2}{(1-8x+32x^2)^2}\Big)\Big)^k\\ &= \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\Big(\frac{x(1-4x)(1-8x)}{(1-8x+32x^2)^2}\Big)^{2k} \pmod p.\end{aligned}$$

By [25, Theorem 2.6], for $n \not\equiv 0, -16 \pmod p$,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{S_k}{(n+16)^k} \equiv \Big(\frac{n(n+16)}{p}\Big)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \pmod p.$$

Set $n = \frac{(1-8x+32x^2)^2}{x(1-4x)(1-8x)}$. Then

$$\frac{1}{n+16} = \frac{x(1-4x)(1-8x)}{(1-32x^2)^2} \quad \text{and} \quad n(n+16) = \Big(\frac{(1-8x+32x^2)(1-32x^2)}{x(1-4x)(1-8x)}\Big)^2.$$

Hence

$$\Big(\sum_{k=0}^{p-1} S_k x^k\Big)^2 \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{n^{2k}} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}S_k\Big(\frac{x(1-4x)(1-8x)}{(1-32x^2)^2}\Big)^k \pmod p$$

as claimed.

Theorem 3.2. *Suppose that $p>3$ is a prime and $x\in\mathbb{Z}_p$ with $x^2+1\not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1}S_k\Big(\frac{x+1}{8}\Big)^k \equiv -\Big(\frac{-3}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-3(x^4-x^2+1)n+(x^2+1)(x^2-2)(2x^2-1)}{p}\Big) \pmod p.$$

Proof. Since $(\frac{-2}{p})=(-1)^{[\frac p4]}$, substituting x with $\frac{x+1}{8}$ in (3.2) yields

$$\sum_{k=0}^{p-1}S_k\Big(\frac{x+1}{8}\Big)^k\equiv\Big(\frac{-(x^2+1)}{p}\Big)\sum_{k=0}^{[p/4]}\binom{2k}{k}\binom{4k}{2k}\Big(\frac{x^2}{16(x^2+1)^2}\Big)^k \pmod p.$$

Set $t=1-\frac{8x^2}{(x^2+1)^2}$. Then $\frac{1-t}{128}=\frac{x^2}{16(x^2+1)^2}$. Also,

$$\frac32(3t+5)=\frac{12}{(x^2+1)^2}(x^4-x^2+1)\quad\text{and}\quad 9t+7=\frac{8(x^2-2)(2x^2-1)}{(x^2+1)^2}.$$

From the above and [23, Lemma 2.2 and Theorem 2.1] we deduce that

$$\begin{aligned}
&\sum_{k=0}^{p-1}S_k\Big(\frac{x+1}{8}\Big)^k\\
&\equiv\Big(\frac{-x^2-1}{p}\Big)\sum_{k=0}^{[p/4]}\binom{2k}{k}\binom{4k}{2k}\Big(\frac{1-t}{128}\Big)^k\equiv\Big(\frac{-x^2-1}{p}\Big)P_{[\frac p4]}(t)\\
&\equiv-\Big(\frac6p\Big)\Big(\frac{-x^2-1}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-\frac32(3t+5)n+9t+7}{p}\Big)\\
&=-\Big(\frac{-6(x^2+1)}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-\frac{12}{(x^2+1)^2}(x^4-x^2+1)n+\frac{8(x^2-2)(2x^2-1)}{(x^2+1)^2}}{p}\Big)\\
&=-\Big(\frac{-6(x^2+1)}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{(\frac{2n}{x^2+1})^3-\frac{12}{(x^2+1)^2}(x^4-x^2+1)\cdot\frac{2n}{x^2+1}+\frac{8(x^2-2)(2x^2-1)}{(x^2+1)^2}}{p}\Big)\\
&=-\Big(\frac{-3}{p}\Big)\sum_{n=0}^{p-1}\Big(\frac{n^3-3(x^4-x^2+1)n+(x^2+1)(x^2-2)(2x^2-1)}{p}\Big) \pmod p.
\end{aligned}$$

This proves the theorem.

Theorem 3.3. *Let p be an odd prime. Then*

$$S_{p-1}\equiv(-1)^{\frac{p-1}{2}}(1+5(2^{p-1}-1)) \pmod{p^2}.$$

Proof. For $m=1,2,\ldots,p-1$ we see that

$$\begin{aligned}
\binom{p-1}{m}&=\frac{(p-1)(p-2)\cdots(p-m)}{m!}\equiv\frac{(-1)(-2)\cdots(-m)}{m!}\Big(1+p\sum_{k=1}^m\frac{1}{-k}\Big)\\
&=(-1)^m(1-pH_m) \pmod{p^2}.
\end{aligned}\tag{3.3}$$

Thus,

$$\begin{aligned}S_{p-1} &= \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \binom{p-1}{2k} 4^{p-1-2k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 (1-pH_{2k})4^{p-1-2k} \\ &= 4^{p-1}\Big(\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} - p\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} H_{2k}\Big) \pmod{p^2}.\end{aligned}$$

It is known that (see [21]) $\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{\frac{p-1}{2}} \pmod{p^2}$. By [32],

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} H_{2k} \equiv \frac{3}{2}(-1)^{\frac{p-1}{2}} H_{\frac{p-1}{2}} \equiv -3(-1)^{\frac{p-1}{2}} \frac{2^{p-1}-1}{p} \pmod{p}.$$

Thus,

$$\begin{aligned}S_{p-1} &\equiv 4^{p-1}\Big(\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} - p\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} H_{2k}\Big) \\ &\equiv 4^{p-1}\cdot(-1)^{\frac{p-1}{2}}\big(1+3(2^{p-1}-1)\big) \pmod{p^2}\end{aligned}$$

Observe that $4^{p-1} = (2^{p-1}-1+1)^2 \equiv 1+2(2^{p-1}-1) \pmod{p^2}$. We then obtain

$$(-1)^{\frac{p-1}{2}} S_{p-1} \equiv (1+2(2^{p-1}-1))(1+3(2^{p-1}-1)) \equiv 1+5(2^{p-1}-1) \pmod{p^2}.$$

This is the result.

Theorem 3.4. *Suppose that $p > 3$ is a prime. Then*

$$\sum_{n=1}^{p-1} \frac{nS_n}{4^n} \equiv -1 \pmod{p^2}.$$

Proof. It is clear that

$$\sum_{n=1}^{p-1} \frac{nS_n}{4^n} = \sum_{n=0}^{p-1}\sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{n}{2k} \frac{n}{4^{2k}} = \sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{1}{16^k} \sum_{n=2k}^{p-1} n\binom{n}{2k}.$$

Since

$$\begin{aligned}\sum_{n=m}^{p-1} n\binom{n}{m} &= (m+1)\sum_{n=m}^{p-1}\binom{n+1}{m+1} - \sum_{n=m}^{p-1}\binom{n}{m} \\ &= (m+1)\binom{p+1}{m+2} - \binom{p}{m+1} = \Big(\frac{m+1}{m+2}(p+1)-1\Big)\binom{p}{m+1},\end{aligned}$$

and $\binom{p}{m+1} = \frac{p}{m+1}\binom{p-1}{m}$ we see that

$$\sum_{n=m}^{p-1} n\binom{n}{m} = \frac{(m+1)p^2-p}{(m+1)(m+2)}\binom{p-1}{m} \tag{3.4}$$

and so

$$\sum_{n=2k}^{p-1} n\binom{n}{2k} = \frac{(2k+1)p^2-p}{(2k+1)(2k+2)}\binom{p-1}{2k} \equiv -\frac{p}{(2k+1)(2k+2)} \pmod{p^2}$$

for $k=0,1,\ldots,\frac{p-1}{2}$. Thus, noting that $p \mid \binom{2k}{k}$ for $\frac{p}{2}<k<p$ we obtain

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{nS_n}{4^n} &\equiv \sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{1}{16^k}\sum_{n=2k}^{p-1} n\binom{n}{2k} \\ &\equiv -\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{p}{16^k(2k+1)(2k+2)} \\ &\equiv -\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{1}{16^k}\Big(\frac{p}{2k+1}-\frac{p}{2(k+1)}\Big) \pmod{p^2}.\end{aligned}$$

By [21, Lemma 2.2],

$$\binom{\frac{p-1}{2}+k}{2k} \equiv \binom{2k}{k}\frac{1}{(-16)^k} \pmod{p^2} \quad \text{for} \quad k=0,1,\ldots,\frac{p-1}{2}.$$

Thus, taking $n=\frac{p-1}{2}$ and $x=1,\frac{1}{2}$ in (2.2) we deduce that
(3.5)

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{1}{16^k(k+1)} \equiv 0 \pmod{p^2}, \quad \sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{p}{16^k(2k+1)} \equiv 1 \pmod{p^2}.$$

Therefore,

$$\sum_{n=1}^{p-1}\frac{nS_n}{4^n} \equiv -\sum_{k=0}^{(p-1)/2}\binom{2k}{k}^2\frac{1}{16^k}\Big(\frac{p}{2k+1}-\frac{p}{2(k+1)}\Big) \equiv -1 \pmod{p^2}.$$

This proves the theorem.

Theorem 3.5. *Suppose that $p>3$ is a prime. Then*

$$\sum_{k=1}^{p-1}\frac{S_k}{4^k k} \equiv 2q_p(2) - p(q_p(2)^2 + 2(-1)^{\frac{p-1}{2}}E_{p-3}) \pmod{p^2}.$$

Proof. It is clear that

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{S_n}{4^n n} &= \sum_{n=1}^{p-1}\sum_{k=0}^{[n/2]}\binom{2k}{k}^2\binom{n}{2k}\frac{1}{4^{2k}n} = \sum_{n=1}^{p-1}\Big(\frac{1}{n}+\sum_{k=1}^{[n/2]}\binom{2k}{k}^2\binom{n}{2k}\frac{1}{4^{2k}n}\Big) \\ &= \sum_{n=1}^{p-1}\frac{1}{n} + \sum_{k=1}^{p-1}\binom{2k}{k}^2\frac{1}{4^{2k}}\sum_{n=2k}^{p-1}\binom{n}{2k}\frac{1}{n} \\ &= H_{p-1} + \sum_{k=1}^{p-1}\binom{2k}{k}^2\frac{1}{4^{2k}\cdot 2k}\sum_{n=2k}^{p-1}\binom{n-1}{2k-1} = H_{p-1} + \frac{1}{2}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{16^k k}\binom{p-1}{2k}.\end{aligned}$$

It is well known that $H_{p-1} \equiv 0 \pmod{p^2}$. By (3.3), $\binom{p-1}{2k} \equiv 1 - pH_{2k} \pmod{p^2}$ for $k \le (p-1)/2$. Also, $p \mid \binom{2k}{k}$ for $p/2 < k < p$. Thus, from the above we obtain

$$\sum_{n=1}^{p-1} \frac{S_n}{4^n n} \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k}(1 - pH_{2k}) \pmod{p^2}.$$

By [33], $\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k} \equiv -2H_{\frac{p-1}{2}} \pmod{p^3}$. By [12], $H_{\frac{p-1}{2}} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}$. Thus,

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k} \equiv -2(-2q_p(2) + pq_p(2)^2) = 4q_p(2) - 2pq_p(2)^2 \pmod{p^2}.$$

Since $p \mid \binom{2k}{k}$ for $\frac{p}{2} < k < p$, from [15, (1.6) and Remark 1.1],

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k} H_{2k} \equiv \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k k} H_{2k} \equiv 4(-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p}.$$

Thus,

$$\begin{aligned}\sum_{n=1}^{p-1} \frac{S_n}{4^n n} &\equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k}(1 - pH_{2k}) = \frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k k} - \frac{p}{2} \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k k} H_{2k} \\ &\equiv 2q_p(2) - pq_p(2)^2 - 2(-1)^{\frac{p-1}{2}} pE_{p-3} \pmod{p^2}\end{aligned}$$

as claimed.

Finally we pose the following conjecture:

Conjecture 3.1. *Let $p > 5$ be a prime. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1} \binom{2k}{k} \frac{21k+8}{25^k} S_k \equiv 8p \pmod{p^2}\\
&\sum_{k=0}^{p-1} \binom{2k}{k} \frac{15k+8}{7^k} S_k \equiv 8\Big(\frac{p}{7}\Big)p \pmod{p^2} \quad \textit{for} \quad p \ne 7,\\
&\sum_{k=0}^{p-1} \binom{2k}{k} \frac{6k+1}{64^k} S_k \equiv (-1)^{\frac{p-1}{2}} p \pmod{p^2},\\
&\sum_{k=0}^{p-1} \binom{2k}{k} \frac{3k+1}{(-32)^k} S_k \equiv (-1)^{[\frac{p}{4}]} p \pmod{p^2},\\
&\sum_{k=0}^{p-1} \binom{2k}{k} \frac{30k+7}{(-128)^k} S_k \equiv 7\Big(\frac{-1}{p}\Big)p \pmod{p^2},\\
&\sum_{k=0}^{p-1} \binom{2k}{k} \frac{6k+1}{160^k} S_k \equiv \Big(\frac{-5}{p}\Big)p \pmod{p^2},\\
&\sum_{k=0}^{p-1} \binom{2k}{k} \frac{7k+1}{800^k} S_k \equiv \Big(\frac{-6}{p}\Big)p \pmod{p^2},\end{aligned}$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{70k+11}{(-768)^k}S_k\equiv 11\Big(\frac{-1}{p}\Big)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{462k+61}{1600^k}S_k\equiv 61\Big(\frac{-1}{p}\Big)p \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{165k+23}{(-1568)^k}S_k\equiv 23\Big(\frac{-2}{p}\Big)p \pmod{p^2}\quad for\quad p\ne 7,$$
$$\sum_{k=0}^{p-1}\binom{2k}{k}\frac{2310k+193}{156832^k}S_k\equiv 193\Big(\frac{-29}{p}\Big)p \pmod{p^2}\quad for\quad p\ne 13,29.$$

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