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Transformation formulas for the number of representations of n by linear combinations of four triangular numbers

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Abstract

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. For $a, b, c, d, n \in \mathbb{Z}^+$ let $t(a, b, c, d; n)$ be the number of representations of n by $ax(x+1)/2 + by(y+1)/2 + cz(z+1)/2 + dw(w+1)/2$ $(x, y, z, w \in \mathbb{Z})$. Using theta function identities we prove 13 transformation formulas for $t(a, b, c, d; n)$, and evaluate $t(2, 3, 3, 8; n)$, $t(1, 1, 6, 24; n)$ and $t(1, 1, 6, 8; n)$.

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. For $a, b, c, d \in \mathbb{Z}^+$ and $n \in \{0, 1, 2, \dots\}$ let $N(a, b, c, d; n)$ be the number of representations of n by $ax^2 + by^2 + cz^2 + dw^2$, where $x, y, z, w \in \mathbb{Z}$. The famous four squares theorem states that every positive integer is the sum of four squares, that is $N(1, 1, 1, 1) > 0$. In 1828, Jacobi showed that

$$N(1,1,1,1;n) = 8 \sum_{d \mid n,\, 4 \nmid d} d.$$

From 1859 to 1866, Liouville made about 90 conjectures on the evaluation of $N(a, b, c, d; n)$. Most of these conjectures have been proved (see Cooper's survey paper [4], Dickson's historical comments [5] and Williams' book [11]).

The numbers $x(x+1)/2$ with $x \in \mathbb{Z}$ are called triangular numbers. For $a, b, c, d \in \mathbb{Z}^+$ and $n \in \{0, 1, 2, \dots\}$, let $t(a, b, c, d; n)$ be the number of representations of n by $ax(x+1)/2 + by(y+1)/2 + cz(z+1)/2 + dw(w+1)/2$ with $x, y, z, w \in \mathbb{Z}$. In 1832, Legendre stated that

$$t(1,1,1,1;n) = 16\sigma(2n+1),$$

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where $\sigma(m)$ is the sum of positive divisors of m. In 2003, Williams [10] showed that $t(1,1,2,2;n) = 4\sigma(4n+3)$. In 2009, Cooper [4] determined $t(a,b,c,d;n)$ for $(a,b,c,d) = (1,1,1,3)$, $(1,3,3,3)$, $(1,2,2,3)$, $(1,3,6,6)$, $(1,3,4,4)$, $(1,1,2,6)$ and $(1,3,12,12)$. In [8], Wang and Sun obtained explicit formulas for $t(a,b,c,d;n)$ in the cases $(a,b,c,d) = (1,2,2,4)$, $(1,2,4,4)$, $(1,1,4,4)$, $(1,4,4,4)$, $(1,3,3,9)$, $(1,1,9,9)$, $(1,9,9,9)$, $(1,1,1,9)$, $(1,3,9,9)$ and $(1,1,3,9)$. In [9] Wang and Sun determined $t(a,b,c,d;n)$ for $(a,b,c,d) = (1,1,2,8)$, $(1,1,2,16)$, $(1,2,3,6)$, $(1,3,4,12)$, $(1,1,3,4)$, $(1,1,5,5)$, $(1,5,5,5)$, $(1,3,3,12)$, $(1,1,1,12)$, $(1,1,3,12)$ and $(1,3,3,4)$, and in [6] the author determined $t(1,3,3,6;n)$, $t(1,1,8,8;n)$ and $t(1,1,4,8;n)$.

For $a,b,c,d \in \mathbb{Z}^+$ and $j = 1,2,3$ let i_j be the number of elements in $\{a,b,c,d\}$ which are equal to j. In 2005, Adiga et al.[1] showed that for $a+b+c+d \in \{5,6,7\}$,

$$t(a,b,c,d;n) = \frac{2}{2+\frac{i_1(i_1-1)}{2}i_2 + i_1 i_3} N(a,b,c,d;8n+a+b+c+d).$$

In 2008, Baruah et al. [2] proved that for $a+b+c+d = 8$,

$$t(a,b,c,d;n) = \frac{2}{2+\frac{i_1(i_1-1)}{2}i_2 + i_1 i_3} (N(a,b,c,d;8n+8) - N(a,b,c,d;2n+2)).$$

In [6], the author stated that for $a,b,n \in \mathbb{Z}^+$ with $2 \nmid a$,

$$t(a,a,2a,4b;4n+3a) = 4t(a,2a,4a,b;n), \quad t(a,a,6a,4b;4n+3a) = 2t(a,a,6a,b;n),$$
$$t(a,a,8a,2b;2n) = t(a,2a,2a,b;n), \quad t(a,a,8a,2b;2n+a) = 2t(a,4a,4a,b;n).$$

Section 2 of this paper is devoted to some definitions and preliminary facts. In Section 3, using theta function identities we prove 13 transformation formulas for $t(a,b,c,d;n)$ in Theorems 3.1-3.3. In Section 4, we use some of the transformation formulas in Section 3 to completely determine $t(2,3,3,8;n)$, $t(1,1,6,24;n)$ and $t(1,1,6,8;n)$ for any positive integer n.

2. Some definitions and preliminary facts

The definition of $t(a,b,c,d;n)$ can be generalised to $t(a_1,\ldots,a_k;n)$. We write $\mathbb{N} = \{0,1,2,\ldots\}$. For $n \in \mathbb{N}$ and $a_1,a_2,\ldots,a_k \in \mathbb{Z}^+$ $(k \ge 2)$, set

$$\begin{aligned} &t(a_1,a_2,\ldots,a_k;n) \\ &= \Big|\Big\{(x_1,\ldots,x_k) \in \mathbb{Z}^k \;\Big|\; n = a_1\frac{x_1(x_1+1)}{2} + a_2\frac{x_2(x_2+1)}{2} + \cdots + a_k\frac{x_k(x_k+1)}{2}\Big\}\Big|. \end{aligned}$$

Since $\frac{(-x-1)(-x)}{2} = \frac{x(x+1)}{2}$ and $8 \cdot \frac{x(x+1)}{2} = (2x+1)^2 - 1$, it is easy to see that

(2.1)

$$\begin{aligned} &t(a_1,a_2,\ldots,a_k;n) \\ &= 2^k\Big|\Big\{(x_1,\ldots,x_k) \in \mathbb{N}^k \;\Big|\; n = a_1\frac{x_1(x_1+1)}{2} + a_2\frac{x_2(x_2+1)}{2} + \cdots + a_k\frac{x_k(x_k+1)}{2}\Big\}\Big| \\ &= \Big|\Big\{(x_1,\ldots,x_k) \in \mathbb{Z}^k \;\Big|\; 8n+a_1+\cdots+a_k = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2,\ 2 \nmid x_1\cdots x_k\Big\}\Big| \\ &= 2^k\Big|\Big\{(x_1,\ldots,x_k) \in \mathbb{N}^k \;\Big|\; 8n+a_1+\cdots+a_k = a_1x_1^2 + a_2x_2^2 + \cdots + a_kx_k^2,\ 2 \nmid x_1\cdots x_k\Big\}\Big|. \end{aligned}$$

Let $\varphi(q)$ and $\psi(q)$ be Ramanujan's theta functions defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \ (|q| < 1).$$

Then clearly

$$\sum_{n=0}^{\infty} t(a_1, a_2, \dots, a_k; n) q^n = 2^k \psi(q^{a_1})\psi(q^{a_2}) \cdots \psi(q^{a_k}) \quad (|q| < 1). \tag{2.2}$$

There are many identities involving $\varphi(q)$ and $\psi(q)$. From [2, Lemma 4.1] or [3], for $|q| < 1$,

$$\psi(q)^2 = \varphi(q)\psi(q^2), \tag{2.3}$$

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8) = \varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8), \tag{2.4}$$

$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2 = \varphi(q^4)^2 + 4q^2\psi(q^8)^2 + 4q\psi(q^4)^2,$$

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}). \tag{2.5}$$

Following [4], let $\{g_1(n)\}$ be given by

$$q\psi(q^6) \prod_{k=1}^{\infty} (1 - q^{2k})^3 = \sum_{n=1}^{\infty} g_1(n) q^n.$$

Using Jacobi's identity

$$\prod_{k=1}^{\infty} (1 - q^k)^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{\frac{k(k+1)}{2}} \quad (|q| < 1)$$

we deduce that

$$\begin{aligned} q\psi(q^6) \prod_{k=1}^{\infty} (1 - q^{2k})^3 &= q\Big(\sum_{m=0}^{\infty} q^{3m(m+1)}\Big)\Big(\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)}\Big) \\ &= \Big(\sum_{m=0}^{\infty} q^{3(2m+1)^2/4}\Big)\Big(\sum_{k=0}^{\infty} (-1)^k (2k+1) q^{(2k+1)^2/4}\Big) \\ &= \sum_{n=0}^{\infty} \Big(\sum_{\substack{k,m \in \{0,1,2,\dots\} \\ 4n = (2k+1)^2 + 3(2m+1)^2}} (-1)^k (2k+1)\Big) q^n \\ &= \sum_{n=0}^{\infty} \Big(\sum_{\substack{a,b \in \mathbb{Z}^+, 2 \nmid a \\ 4n = a^2 + 3b^2}} (-1)^{(a-1)/2} a\Big) q^n. \end{aligned}$$

Hence,

$$g_1(n) = \sum_{\substack{a,b \in \mathbb{Z}^+, 2 \nmid a \\ 4n = a^2 + 3b^2}} (-1)^{(a-1)/2} a. \tag{2.6}$$

Let $\{a(n)\}$ be given by

$$q\prod_{k=1}^{\infty}(1-q^{2k})(1-q^{4k})(1-q^{6k})(1-q^{12k})=\sum_{n=1}^{\infty}a(n)q^n \quad (|q|<1).$$

It is well known that $a(n)$ is a multiplicative function concerned with a weight-2 newform (see, for example, [7]). For $a\in\mathbb{Z}$ and $m\in\{1,3,5,\ldots\}$, as usual, $(\frac{a}{m})$ denotes the Jacobi symbol.

3. New transformation formulas for $t(a,b,c,d;n)$

In this section we present 13 transformation formulas for $t(a,b,c,d;n)$, where $a,b,c,d,n\in\mathbb{Z}^+$.

Theorem 3.1. *Let $a,b,c\in\mathbb{Z}^+$ and $2\nmid a$. For $n\in\mathbb{Z}^+$ we have*

$$t(a,a,2b,2c;2n+a)=2t(a,4a,b,c;n), \tag{3.1}$$

$$t(a,3a,4a,2b;2n+a)=t(a,a,6a,b;n), \tag{3.2}$$

$$t(a,3a,12a,2b;2n)=t(2a,3a,3a,b;n). \tag{3.3}$$

Proof. Using (2.2)-(2.4),

$$\begin{aligned}&\sum_{n=0}^{\infty}t(a,a,2b,2c;n)q^n=16\psi(q^a)^2\psi(q^{2b})\psi(q^{2c})\\&=16\varphi(q^a)\psi(q^{2a})\psi(q^{2b})\psi(q^{2c})=16(\varphi(q^{4a})+2q^a\psi(q^{8a}))\psi(q^{2a})\psi(q^{2b})\psi(q^{2c}).\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}t(a,a,2b,2c;2n+a)q^{2n+a}=32q^a\psi(q^{8a})\psi(q^{2a})\psi(q^{2b})\psi(q^{2c})$$

and so

$$\sum_{n=0}^{\infty}t(a,a,2b,2c;2n+a)q^n=32\psi(q^a)\psi(q^{4a})\psi(q^b)\psi(q^c)=2\sum_{n=0}^{\infty}t(a,4a,b,c;n)q^n,$$

which yields (3.1). Also, appealing to (2.5),

$$\begin{aligned}&\sum_{n=0}^{\infty}t(a,3a,4a,2b;n)q^n\\&=16\psi(q^a)\psi(q^{3a})\psi(q^{4a})\psi(q^{2b})=16(\varphi(q^{6a})\psi(q^{4a})+q^a\varphi(q^{2a})\psi(q^{12a}))\psi(q^{4a})\psi(q^{2b}).\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}t(a,3a,4a,2b;2n+a)q^{2n+a}=16q^a\varphi(q^{2a})\psi(q^{4a})\psi(q^{12a})\psi(q^{2b})=16q^a\psi(q^{2a})^2\psi(q^{12a})\psi(q^{2b})$$

and so

$$\sum_{n=0}^{\infty}t(a,3a,4a,2b;2n+a)q^n=16\psi(q^a)^2\psi(q^{6a})\psi(q^b)=\sum_{n=0}^{\infty}t(a,a,6a,b;n)q^n.$$

This yields (3.2). In view of (2.5),

$$\sum_{n=0}^{\infty} t(a,3a,12a,2b;n)q^n$$
$$= 16\psi(q^a)\psi(q^{3a})\psi(q^{12a})\psi(q^{2b}) = 16(\varphi(q^{6a})\psi(q^{4a}) + q^a\varphi(q^{2a})\psi(q^{12a}))\psi(q^{12a})\psi(q^{2b}).$$

Therefore

$$\sum_{n=0}^{\infty} t(a,3a,12a,2b;2n)q^{2n} = 16\varphi(q^{6a})\psi(q^{4a})\psi(q^{12a})\psi(q^{2b}) = 16\psi(q^{6a})^2\psi(q^{4a})\psi(q^{2b})$$

and so

$$\sum_{n=0}^{\infty} t(a,3a,12a,2b;2n)q^n = 16\psi(q^{2a})\psi(q^{3a})^2\psi(q^b) = \sum_{n=0}^{\infty} t(2a,3a,3a,b;n)q^n,$$

which gives (3.3). The proof is now complete.

Theorem 3.2. *Suppose that $a,b,c \in \mathbb{Z}^+$ and $2 \nmid a$. For $n \in \mathbb{Z}^+$,*

$$t(a,3a,4b,4c;4n+3a) = 2t(3a,4a,b,c;n), \tag{3.4}$$
$$t(a,3a,4b,4c;4n+6a) = 2t(a,12a,b,c;n), \tag{3.5}$$
$$t(a,3a,48a,4b;4n) = t(a,6a,6a,b;n), \tag{3.6}$$
$$t(2a,3a,3a,4b;4n+3a) = 2t(2a,3a,3a,b;n). \tag{3.7}$$

Proof. By appealing to (2.5) and (2.4),

$$\sum_{n=0}^{\infty} t(a,3a,4b,4c;n)q^n$$
$$= 16\psi(q^a)\psi(q^{3a})\psi(q^{4b})\psi(q^{4c}) = 16(\varphi(q^{6a})\psi(q^{4a}) + q^a\varphi(q^{2a})\psi(q^{12a}))\psi(q^{4b})\psi(q^{4c})$$
$$= 16((\varphi(q^{24a}) + 2q^{6a}\psi(q^{48a}))\psi(q^{4a}) + q^a(\varphi(q^{8a}) + 2q^{2a}\psi(q^{16a}))\psi(q^{12a}))\psi(q^{4b})\psi(q^{4c}).$$

Thus,

$$\sum_{n=0}^{\infty} t(a,3a,4b,4c;4n+6a)q^{4n+6a} = 32q^{6a}\psi(q^{48a})\psi(q^{4a})\psi(q^{4b})\psi(q^{4c}),$$
$$\sum_{n=0}^{\infty} t(a,3a,4b,4c;4n+3a)q^{4n+3a} = 32q^{3a}\psi(q^{16a})\psi(q^{12a})\psi(q^{4b})\psi(q^{4c}),$$
$$\sum_{n=0}^{\infty} t(a,3a,4b,48a;4n)q^{4n} = 16\varphi(q^{24a})\psi(q^{4a})\psi(q^{4b})\psi(q^{48a})$$

and so

$$\sum_{n=0}^{\infty} t(a,3a,4b,4c;4n+6a)q^n = 32\psi(q^a)\psi(q^{12a})\psi(q^b)\psi(q^c) = 2\sum_{n=0}^{\infty} t(a,12a,b,c;n)q^n,$$

$$\sum_{n=0}^{\infty} t(a,3a,4b,4c;4n+3a)q^n = 32\psi(q^{4a})\psi(q^{3a})\psi(q^b)\psi(q^c) = 2\sum_{n=0}^{\infty} t(3a,4a,b,c;n)q^n,$$
$$\sum_{n=0}^{\infty} t(a,3a,4b,48a;4n)q^n = 16\varphi(q^{6a})\psi(q^{12a})\psi(q^a)\psi(q^b) = 16\psi(q^a)\psi(q^{6a})^2\psi(q^b),$$

which yields (3.4)-(3.6). To prove (3.7), we appeal to (3.1) and (3.3) to see that

$$t(2a,3a,3a,4b;4n+3a) = 2t(a,3a,12a,2b;2n) = 2t(2a,3a,3a,b;n).$$

Theorem 3.3. *Suppose that $a, b \in \mathbb{Z}^+$ with $2 \nmid a$. For $n \in \mathbb{Z}^+$,*

$$t(2a,3a,3a,8b;8n+9a) = 4t(3a,3a,4a,b;n), \tag{3.8}$$
$$t(a,a,6a,8b;8n+13a) = 4t(a,a,12a,b;n), \tag{3.9}$$
$$t(a,a,6a,8b;8n+4a) = 2t(a,a,3a,b;n), \tag{3.10}$$
$$t(a,a,6a,8b;8n+6a) = 4t(2a,2a,3a,b;n), \tag{3.11}$$
$$t(2a,3a,3a,8b;8n+12a) = 4t(a,6a,6a,b;n), \tag{3.12}$$
$$t(2a,3a,3a,8b;8n+6a) = 2t(a,3a,3a,b;n). \tag{3.13}$$

Proof. By (3.1), (3.4) and (3.5),

$$t(2a,3a,3a,8b;8n+9a) = 2t(a,3a,12a,4b;4n+3a) = 4t(3a,3a,4a,b;n),$$
$$t(a,a,6a,8b;8n+13a) = 2t(a,3a,4a,4b;4n+6a) = 4t(a,a,12a,b;n).$$

This proves (3.8) and (3.9). By (2.3), (2.4) and (2.5),

$$\begin{aligned}
&\sum_{n=0}^{\infty} t(a,a,6a,8b;n)q^n \\
&= 16\psi(q^a)^2\psi(q^{6a})\psi(q^{8b}) = 16\varphi(q^a)\psi(q^{2a})\psi(q^{6a})\psi(q^{8b}) \\
&= 16(\varphi(q^{4a}) + 2q^a\psi(q^{8a}))(\varphi(q^{12a})\psi(q^{8a}) + q^{2a}\varphi(q^{4a})\psi(q^{24a}))\psi(q^{8b}) \\
&= 16(\varphi(q^{16a}) + 2q^{4a}\psi(q^{32a}) + 2q^a\psi(q^{8a})) \\
&\quad \times (\varphi(q^{48a})\psi(q^{8a}) + 2q^{12a}\psi(q^{96a})\psi(q^{8a}) + q^{2a}\varphi(q^{16a})\psi(q^{24a}) + 2q^{6a}\psi(q^{24a})\psi(q^{32a}))\psi(q^{8b}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty} t(a,a,6a,8b;8n+6a)q^{8n+6a} \\
&= 16\psi(q^{8b})(\varphi(q^{16a})\cdot 2q^{6a}\psi(q^{24a})\psi(q^{32a}) + 2q^{4a}\psi(q^{32a})\cdot q^{2a}\varphi(q^{16a})\psi(q^{24a})) \\
&= 64q^{6a}\varphi(q^{16a})\psi(q^{32a})\psi(q^{24a})\psi(q^{8b}) = 64q^{6a}\psi(q^{16a})^2\psi(q^{24a})\psi(q^{8b})
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} t(a,a,6a,8b;8n+4a)q^{8n+4a} \\
&= 16\cdot 2q^{4a}(\varphi(q^{48a})\psi(q^{32a}) + q^{8a}\varphi(q^{16a})\psi(q^{96a}))\psi(q^{8a})\psi(q^{8b})
\end{aligned}$$

$$= 32q^{4a}\psi(q^{8a})^2\psi(q^{24a})\psi(q^{8b}).$$

It then follows that

$$\sum_{n=0}^{\infty} t(a,a,6a,8b;8n+6a)q^n = 64\psi(q^{2a})^2\psi(q^{3a})\psi(q^b) = 4\sum_{n=0}^{\infty} t(2a,2a,3a,b;n)q^n,$$
$$\sum_{n=0}^{\infty} t(a,a,6a,8b;8n+4a)q^n = 32\psi(q^{a})^2\psi(q^{3a})\psi(q^b) = 2\sum_{n=0}^{\infty} t(a,a,3a,b;n)q^n,$$

which yields (3.10) and (3.11). By (2.3), (2.4) and (2.5),

$$\begin{aligned}
&\sum_{n=0}^{\infty} t(2a,3a,3a,8b;n)q^n \\
&= 16\psi(q^{2a})\psi(q^{3a})^2\psi(q^{8b}) = 16\varphi(q^{3a})\psi(q^{2a})\psi(q^{6a})\psi(q^{8b}) \\
&= 16(\varphi(q^{12a})+2q^{3a}\psi(q^{24a}))(\varphi(q^{12a})\psi(q^{8a})+q^{2a}\varphi(q^{4a})\psi(q^{24a}))\psi(q^{8b}).
\end{aligned}$$

Thus

$$\sum_{n=0}^{\infty} t(2a,3a,3a,8b;4n)q^{4n} = 16\varphi(q^{12a})^2\psi(q^{8a})\psi(q^{8b}),$$
$$\sum_{n=0}^{\infty} t(2a,3a,3a,8b;4n+2a)q^{4n+2a} = 16q^{2a}\varphi(q^{4a})\varphi(q^{12a})\psi(q^{24a})\psi(q^{8b})$$

and so

$$\begin{aligned}
&\sum_{n=0}^{\infty} t(2a,3a,3a,8b;4n)q^n = 16\varphi(q^{3a})^2\psi(q^{2a})\psi(q^{2b}) \\
&\quad = 16(\varphi(q^{6a})^2+4q^{3a}\psi(q^{12a})^2)\psi(q^{2a})\psi(q^{2b}), \\
&\sum_{n=0}^{\infty} t(2a,3a,3a,8b;4n+2a)q^n = 16\varphi(q^{a})\varphi(q^{3a})\psi(q^{6a})\psi(q^{2b}) \\
&\quad = 16(\varphi(q^{4a})+2q^{a}\psi(q^{8a}))(\varphi(q^{12a})+2q^{3a}\psi(q^{24a}))\psi(q^{6a})\psi(q^{2b}).
\end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} t(2a,3a,3a,8b;4(2n+3a))q^{2n+3a} = 64q^{3a}\psi(q^{2a})\psi(q^{12a})^2\psi(q^{2b}),$$
$$\begin{aligned}
&\sum_{n=0}^{\infty} t(2a,3a,3a,8b;4(2n+a)+2a)q^{2n+a} \\
&\quad = 32q^{a}(\varphi(q^{12a})\psi(q^{8a})+q^{2a}\varphi(q^{4a})\psi(q^{24a}))\psi(q^{6a})\psi(q^{2b}) = 32q^{a}\psi(q^{2a})\psi(q^{6a})^2\psi(q^{2b}).
\end{aligned}$$

This yields

$$\sum_{n=0}^{\infty} t(2a,3a,3a,8b;8n+12a)q^n = 64\psi(q^{a})\psi(q^{6a})^2\psi(q^b) = 4\sum_{n=0}^{\infty} t(a,6a,6a,b;n)q^n,$$

$$\sum_{n=0}^{\infty} t(2a,3a,3a,8b;8n+6a)q^n = 32\psi(q^a)\psi(q^{3a})^2\psi(q^b) = 2\sum_{n=0}^{\infty} t(a,3a,3a,b;n)q^n,$$

which implies (3.12) and (3.13). The proof is now complete.

Remark 3.4 With minor changes in the proofs of Theorems 3.1-3.3, one can obtain corresponding generalised results for $t(a_1,\dots,a_k;n)$. For instance, (3.1), (3.4) and (3.5) can be generalised to

$$\begin{aligned} t(a,a,2b_1,\dots,2b_r;2n+a) &= 2t(a,4a,b_1,\dots,b_r;n),\\ t(a,3a,4b_1,\dots,4b_r;4n+3a) &= 2t(3a,4a,b_1,\dots,b_r;n),\\ t(a,3a,4b_1,\dots,4b_r;4n+6a) &= 2t(a,12a,b_1,\dots,b_r;n), \end{aligned}$$

where $a,b_1,\dots,b_r \in \mathbb{Z}^+$ with $2 \nmid a$.

4. Evaluation of $t(2,3,3,8;n)$, $t(1,1,6,24;n)$ and $t(1,1,6,8;n)$

Theorem 4.1. *Let n be a positive integer.*

(i) *If $2n+5 = 3^{\beta}n_1$ $(3 \nmid n_1)$, then*

$$t(1,1,6,24;2n+1) = t(2,3,3,8;2n+3) = 4(\sigma(n_1) - (-1)^n a(2n+5)).$$

(ii) *If $n+1 = 2^{\alpha}3^{\beta}n_1$ with $2 \nmid n_1$ and $3 \nmid n_1$, then*

$$t(1,1,6,24;2n-2) + t(2,3,3,8;2n) = 2^{\alpha+4}\sigma(n_1).$$

Proof. From (3.1), $t(1,1,6,24;2n+1) = 2t(1,3,4,12;n) = t(2,3,3,8;2n+3)$. If $2n+5 = 3^{\beta}n_1$ with $3 \nmid n_1$, then $t(1,3,4,12;n) = 2(\sigma(n_1) - (-1)^n a(2n+5))$ by [9, Theorem 3.4]. Thus, (i) is true.

Now we prove (ii). Since $\psi(q)^2 = \varphi(q)\psi(q^2) = (\varphi(q^4) + 2q\psi(q^8))\psi(q^2)$,

$$\sum_{n=0}^{\infty} t(1,1,6,24;n)q^n = 16\psi(q)^2\psi(q^6)\psi(q^{24}) = (16\varphi(q^4) + 32q\psi(q^8))\psi(q^2)\psi(q^6)\psi(q^{24}).$$

Thus,

$$\sum_{n=0}^{\infty} t(1,1,6,24;2n)q^{2n} = 16\varphi(q^4)\psi(q^2)\psi(q^6)\psi(q^{24})$$

and so

$$\sum_{n=0}^{\infty} t(1,1,6,24;2n)q^n = 16\varphi(q^2)\psi(q)\psi(q^3)\psi(q^{12}). \tag{4.1}$$

Similarly,

$$\sum_{n=0}^{\infty} t(2,3,3,8;n)q^n = 16\psi(q^2)\psi(q^3)^2\psi(q^8) = (16\varphi(q^{12}) + 32q^3\psi(q^{24}))\psi(q^6)\psi(q^2)\psi(q^8).$$

Hence,
$$\sum_{n=0}^{\infty} t(2,3,3,8;2n)q^{2n} = 16\psi(q^2)\psi(q^6)\psi(q^8)\varphi(q^{12})$$
and so
$$\sum_{n=0}^{\infty} t(2,3,3,8;2n)q^{n} = 16\psi(q)\psi(q^3)\psi(q^4)\varphi(q^6). \tag{4.2}$$
From (4.1), (4.2) and (2.5),
$$\begin{aligned}&\sum_{n=0}^{\infty}(t(2,3,3,8;2n)+t(1,1,6,24;2n-2))q^n\\ &= 16\psi(q)\psi(q^3)(\varphi(q^6)\psi(q^4)+q\varphi(q^2)\psi(q^{12})) = 16\psi(q)^2\psi(q^3)^2.\end{aligned}$$
Hence
$$t(2,3,3,8;2n)+t(1,1,6,24;2n-2) = t(1,1,3,3;n).$$
If $n+1 = 2^{\alpha}3^{\beta}n_1$ with $2 \nmid n_1$ and $3 \nmid n_1$, then $t(1,1,3,3;n) = 2^{\alpha+4}\sigma(n_1)$ by [8, Lemma 4.1]. Thus, (ii) holds and the proof is complete.

Theorem 4.2. *Let n be a positive integer.*
(i) *If $n+1 = 2^{\alpha}3^{\beta}n_1$ with $2 \nmid n_1$ and $3 \nmid n_1$, then*
$$t(2,3,3,8;4n+2) = t(1,1,6,24;4n) = 2^{\alpha+4}\sigma(n_1).$$
(ii) *If $2n+1 = 3^{\beta}n_1$ $(3 \nmid n_1)$, then*
$$\begin{aligned}t(2,3,3,8;4n) &= 8(\sigma(n_1)+a(2n+1)),\\ t(1,1,6,24;4n-2) &= 8(\sigma(n_1)-a(2n+1)).\end{aligned}$$

Proof. If $n+1 = 2^{\alpha}3^{\beta}n_1$ with $2 \nmid n_1$ and $3 \nmid n_1$, from Theorem 4.1,
$$t(1,1,6,24;4n)+t(2,3,3,8;4n+2) = 2^{\alpha+5}\sigma(n_1).$$
By [6, Theorem 2.14], $t(1,1,6,24;4n) = 2^{\alpha+4}\sigma(n_1)$. Thus, $t(2,3,3,8;4n+2) = 2^{\alpha+4}\sigma(n_1)$. This proves (i).

Now we consider (ii). Suppose that $2n+1 = 3^{\beta}n_1$ $(3 \nmid n_1)$. We first assume that n is odd and $n = 2m+1$. By (3.12), $t(2,3,3,8;8m+12) = 4t(1,1,6,6;m)$. This together with [7, Theorem 4.15] yields
$$t(2,3,3,8;8m+4) = 4t(1,1,6,6;m-1) = 8(\sigma(n_1)+a(4m+3)).$$
Now, appealing to Theorem 4.1(ii),
$$t(1,1,6,24;8m+2) = 16\sigma(n_1) - t(2,3,3,8;8m+4) = 8(\sigma(n_1)-a(4m+3)).$$
From now on suppose that n is even and $n = 2m$. From (3.11) (with $a = 1$, $b = 3$ and $n = m-1$) and [7, Theorem 4.15],
$$t(1,1,6,24;8m-2) = 4t(2,2,3,3;m-1) = 8(\sigma(n_1)-a(4m+1)).$$

Now applying Theorem 4.1(ii) gives

$$t(2,3,3,8;8m) = 16\sigma(n_1) - t(1,1,6,24;8m-2) = 8(\sigma(n_1) + a(4m+1)).$$

Putting all the above together proves the theorem.

Theorem 4.3. *Let $n \in \mathbb{Z}^+$.*
(i) *If $n+1 = 2^\alpha 3^\beta n_1$ with $2 \nmid n_1$ and $3 \nmid n_1$, then*

$$t(1,1,6,8;2n) = 2^{\alpha+2}\Big(3^{\beta+1}\Big(\frac{3}{n_1}\Big) + (-1)^{\alpha+\beta+\frac{n_1-1}{2}}\Big)\sum_{d|n_1} d\Big(\frac{3}{d}\Big).$$

(ii) *If $2n+3 = 3^\beta n_1$ with $3 \nmid n_1$, then*

$$t(1,1,6,8;2n+1) = 2\Big(3^{\beta+1}\Big(\frac{3}{n_1}\Big) + (-1)^n\Big)\sum_{d|n_1} d\Big(\frac{3}{d}\Big) - 4\sum_{\substack{a,b\in\mathbb{Z}^+,2\nmid a\\ 8n+12=a^2+3b^2}} (-1)^{(a-1)/2}a.$$

Proof. From [6, Theorem 2.7],

$$t(1,1,6,8;2n) = t(1,2,2,3;n) \quad \text{and} \quad t(1,1,6,8;2n+1) = 2t(1,3,4,4;n).$$

By (2.1) and [4, Theorem 5.4], if $n+1 = 2^\alpha 3^\beta n_1$ with $2 \nmid n_1$ and $3 \nmid n_1$, then

$$\begin{aligned} t(1,2,2,3;n) &= 2^{\alpha+2}\Big(3^{\beta+1} + (-1)^{\alpha+\beta}\Big(\frac{-3}{n_1}\Big)\Big)\sum_{d|n_1} \frac{n_1}{d}\Big(\frac{3}{d}\Big) \\ &= 2^{\alpha+2}\Big(3^{\beta+1} + (-1)^{\alpha+\beta}\Big(\frac{-3}{n_1}\Big)\Big)\sum_{d|n_1} d\Big(\frac{3}{n_1/d}\Big) \\ &= 2^{\alpha+2}\Big(3^{\beta+1}\Big(\frac{3}{n_1}\Big) + (-1)^{\alpha+\beta+(n_1-1)/2}\Big)\sum_{d|n_1} d\Big(\frac{3}{d}\Big). \end{aligned}$$

Thus part(i) is true.

Let us consider part(ii). By (2.1) and [4, Theorem 5.6], if $2n+3 = 3^\beta n_1$ with $3 \nmid n_1$, then

$$t(1,3,4,4;n) = \Big(3^{\beta+1} - (-1)^\beta\Big(\frac{-3}{n_1}\Big)\Big)\sum_{d|n_1} \frac{n_1}{d}\Big(\frac{3}{d}\Big) - 2g_1(2n+3).$$

From the above and (2.6),

$$t(1,1,6,8;2n+1) = 2t(1,3,4,4;n) = 2\Big(3^{\beta+1} - (-1)^\beta\Big(\frac{-3}{n_1}\Big)\Big)\sum_{d|n_1} d\Big(\frac{3}{n_1/d}\Big) - 4g_1(2n+3)$$

$$= 2\Big(3^{\beta+1}\Big(\frac{3}{n_1}\Big) - (-1)^\beta\Big(\frac{-1}{n_1}\Big)\Big)\sum_{d|n_1} d\Big(\frac{3}{d}\Big) - 4\sum_{\substack{a,b\in\mathbb{Z}^+,2\nmid a\\ 8n+12=a^2+3b^2}} (-1)^{(a-1)/2}a.$$

Observe that $(-1)^\beta\big(\frac{-1}{n_1}\big) \equiv 3^\beta n_1 = 2n+3 \equiv (-1)^{n-1} \pmod 4$ and so $-(-1)^\beta\big(\frac{-1}{n_1}\big) = (-1)^n$. We therefore obtain part(ii). The proof is complete.

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